Functional Analysis Tutorial Class 1

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Outline

1. Jensen's inequality and Young's inequality
2. Hölder's inequality and Minkovski inequality
3. Examples of Banach spaces
4. Counterexample

Outline

- **Jensen's inequality**

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  (Finite form) For a real **convex function** φ, numbers $x_1, x_2, \cdots, x_n$ in its domain, and positive weights a_i, Jensen's inequality can be stated as

$$\varphi\left(\frac{1}{\sum a_i}\sum a_i x_i\right) \leq \frac{1}{\sum a_i}\sum a_i \varphi\left(x_i\right)$$

Jensen's inequality and Young's inequality

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  (Measure-theoretic and probabilistic form) Let (Ω, A, μ) be a probability space such that $\mu(\Omega) = 1$ If g is a real valued function which is μ -integrable and φ is a convex function on the real line, then

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu$$

- **Young's inequality**

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  (for products) In standard form, the inequality states that if a, b are nonnegative real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

The equality holds if and only if $a^p = b^q$.

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  The equality holds if and only if $a^p = b^q$.
  LHS=0
  Proof: The claim is certainly true if $a = 0$ or $b = 0$. Therefore, assume $a > 0$ and $b > 0$ in the following.

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$$\ln\left(ta^p + (1-t)b^q\right) \geq t\ln\left(a^p\right) + (1-t)\ln\left(b^q\right) = \ln(a) + \ln(b) = \ln(ab)$$

$= \cdot \ln a$

with the equality holding if and only if $a^p = b^q$.

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with the equality holding if and only if $a^p = b^q$. Young's inequality follows by exponentiating.

Remark : The numbers p, q are said to be **Hölder conjugates** of each other.

Outline

- **Hölder's inequality**

$\frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$

(for the counting measure) If p, q are Hölder conjugates, then

$$\sum |x_i y_i| \le \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |y_i|^q\right)^{\frac{1}{q}}$$

for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$

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  Proof: WLOG, we assume that $\sum |x_i|^p > 0$, $\sum |y_i|^q > 0$. Let

$$a = \frac{|x_i|}{\left(\sum |x_i|^p\right)^{\frac{1}{p}}}, \quad b = \frac{|y_i|}{\left(\sum |y_i|^q\right)^{\frac{1}{q}}}$$

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$$b^q = \left(\frac{|y_i|}{\left(\sum |y_i|^q\right)^{\frac{1}{q}}}\right)^q = \frac{|y_i|^q}{\left(\sum |y_i|^q\right)}$$

Applying Young's inequality $\left(ab \leq \frac{a^p}{p} + \frac{b^q}{q}\right)$, we get

$$\frac{|x_i y_i|}{\left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |y_i|^q\right)^{\frac{1}{q}}} \leq \frac{|x_i|^p}{p \sum |x_i|^p} + \frac{|y_i|^q}{q \sum |y_i|^q}, \quad 1 \leq i \leq n$$

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$\sum |x_i y_i| \leq (\sum |x_i|^p)^{\frac{1}{p}} (\cdot \sum |y_i|^q)^{\frac{1}{q}}$

Summing up over i,

$$\boxed{\frac{\sum |x_i y_i|}{\left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |y_i|^q\right)^{\frac{1}{q}}}} \leq \frac{\cancel{\sum |x_i|^p}}{p \cancel{\sum |x_i|^p}} + \frac{\cancel{\sum |y_i|^q}}{q \cancel{\sum |y_i|^q}} = \frac{1}{p} + \frac{1}{q} = 1$$

Hölder's inequality and Minkovski inequality

- **Minkovski inequality**

- **Minkovski inequality** *(for the counting measure)* For any $p \geq 1$

$$\left(\sum |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}}$$

for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$

- $p=1$,

$$\sum |x_i + y_i| \leq \sum |x_i| + \sum |y_i| \quad \checkmark$$

- $p>1$

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for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$

Proof (of the case $p > 1$): ~~By Hölder inequality,~~

$p = 1 + (p-1)$

$(\ |x_i+y_i| \leq |x_i| + |y_i|$

$$\begin{aligned}
&\sum |x_i + y_i|^p \\
= &\sum |x_i + y_i| |x_i + y_i|^{p-1} \\
\leq &\sum |x_i| |x_i + y_i|^{p-1} + \sum_i |y_i| |x_i + y_i|^{p-1}
\end{aligned}$$

$(Z_i = |x_i + y_i|^{p-1})$

$\sum |x_i \cdot Z_i| \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum |Z_i|^q\right)^{\frac{1}{q}}$

$\sum |y_i Z_i| \leq \left(\sum |y_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum |Z_i|^q\right)^{\frac{1}{q}}$

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for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$

Proof (of the case $p > 1$): By Hölder inequality,

$$\frac{1}{q} + \frac{1}{p} = 1$$

$$\begin{aligned}
&\sum |x_i + y_i|^p \\
&= \sum |x_i + y_i|\,|x_i + y_i|^{p-1} \\
&\leq \sum |x_i|\,|x_i + y_i|^{p-1} + \sum_i |y_i||x_i + y_i|^{p-1} \\
&\leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^p\right)^{\frac{1}{q}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i + y_i|^p\right)^{\frac{1}{q}} \\
&= \left[\left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left(\sum |x_i + y_i|^p\right)^{\frac{1}{q}}
\end{aligned}$$

- **Minkovski inequality** *(for the counting measure)* For any $p \geq 1$

$\frac{1}{p}+\frac{1}{q}=1$

$$\left(\sum |x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}}$$

for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$

Proof (of the case $p>1$): By Hölder inequality,

$$\left(\Sigma|x_i+y_i|^p\right)^{\frac{1}{p}} = \frac{\left(\Sigma|x_i+y_i|^p\right)}{\left(\Sigma|x_i+y_i|^p\right)^{\frac{1}{q}}} \leq \left(\Sigma|x_i|^p\right)^{\frac{1}{p}} + \left(\Sigma|y_i|^p\right)^{\frac{1}{p}}$$

$$\begin{aligned}
&\sum |x_i+y_i|^p \\
&= \sum |x_i+y_i|\,|x_i+y_i|^{p-1} \\
&\leq \sum |x_i|\,|x_i+y_i|^{p-1} + \sum_i |y_i||x_i+y_i|^{p-1} \\
&\leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum |x_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i+y_i|^p\right)^{\frac{1}{q}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}} \left(\sum |x_i+y_i|^p\right)^{\frac{1}{q}}
\end{aligned}$$

Divide both sides by $\left(\sum |x_i+y_i|^p\right)^{\frac{1}{q}}$ and the desired inequality follows.

- **Minkovski inequality**
  (for the counting measure) For any $p \geq 1$,

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  for complex numbers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$
  (for the Lebesgue measure) If Ω is a measurable subset of $\mathbb{R}^n$,with the Lebesgue measure and f, g are measurable complex-valued functions on Ω, then

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$$\left(\int_\Omega |f(x) + g(x)|^p\,dx\right)^{\frac{1}{p}} \leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^p\,dx\right)^{\frac{1}{p}}$$

Outline

Example 1

- $X = C(K)$, where K ~~is a compact subset~~ in $\mathbb{R}^n$. Define
$$\|f\|_X = \sup_{x \in K} |f(x)|.$$
Then $(X, \|\cdot\|_X)$ is a Banach space.

· normed space

· $\{f_n\} \rightarrow f \in C(K)$.

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Let's prove the Completeness. Suppose $\{f_n\}$ is a $\boxed{\text{Cauchy sequence}}$ in X.
Then $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall m, k > N$ and ~~$x \in K$,~~ it holds that

$$\sup_{x \in K} |f_m(x) - f_k(x)| < \varepsilon.$$

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Therefore, for each $x \in K$, $\{f_k(x)\}_k$ is Cauchy in $\mathbb{R}^n$ and thus we can define funciton

$$f(x) := \lim_{k \to \infty} f_k(x), \qquad x \in K.$$

$f_k \to f$ pointwisely

$f_k \xrightarrow{\text{normly}}$ convergent to f

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$k \to \infty$

$$f(x) = \lim_{k\to\infty} f_k(x), \qquad x \in K.$$

Since N is independent of x, we can take $k \to \infty$ and consequently

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$f_m \overset{\text{normly}}{\rightrightarrows} f$,

$$\sup_{x \in K} |f_m(x) - f(x)| < \varepsilon.$$

f_m converges uniformly to f.

If we can prove $f \in C(k)$.

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$$\|f\|_X = \sup_{x \in K} |f(x)|.$$

$\sup_{x \in K} |f_m(x) - f(x)| < \varepsilon$

$\Updownarrow$

$f_m \rightrightarrows f$

Then $(X, \|\cdot\|_X)$ is a Banach space.

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Therefore, for each $x \in K$, $\{f_k(x)\}$ is Cauchy in $\mathbb{R}^n$ and thus we can define funciton

$$f(x) = \lim_{k \to \infty} f_k(x), \qquad x \in K.$$

Since N is independent of x, we can take $k \to \infty$ and consequently

$$\sup_{x \in K} |f_m(x) - f(x)| < \varepsilon.$$

$|f(x) - f(y)| \le |f(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$

f_m converges uniformly to f. Hence $f \in C(K)$ by compactness of K, which finishes the proof.

Outline

Counterexample

- Let $K = [0,1]$. Define a norm $\|f\|_1 := \int_0^1 |f(x)|\,dx$. Then $(\underbrace{\mathcal{C}[0,1]}_{X}, \|\cdot\|_1)$ is not a Banach space.

· $\exists$ Cauchy sequence $\{f_n\}$

$f_n \not\to f$

or

$f_n \to f$, but $f \notin X$.

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  Idea: Find a Cauchy sequence $\{f_n\}$, which converges to a function $f \notin \mathcal{C}[0,1]$.

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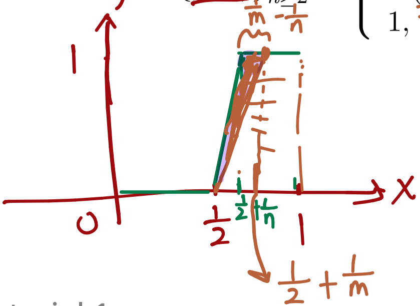
  Idea: Find a Cauchy sequence $\{f_n\}$, which converges to a function $f \notin \mathcal{C}[0,1]$.
  Consider the sequence

$$\{f_n(x)\}_{n\geq 2} = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ n\left(x-\frac{1}{2}\right), & \frac{1}{2} < x \leq \frac{1}{2}+\frac{1}{n} \\ 1, & \frac{1}{2}+\frac{1}{n} < x \leq 1 \end{cases}$$

Handwritten beside the cases: $\frac{1}{2}$; $=0$; $\frac{1}{2}+\frac{1}{n}$ $=1$



$$\|f_n - f_m\| = \frac{1}{2}\left(\frac{1}{m}-\frac{1}{n}\right)$$

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  It's a Cauchy sequence in $(\mathcal{C}[0,1], \|\cdot\|_1)$ since

$$\|f_n - f_m\|_1 = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \to 0 \text{ as } n, m \to \infty$$

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$f \notin C[0,1]$

Let

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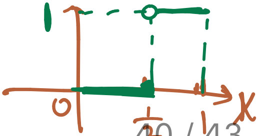


Counterexample

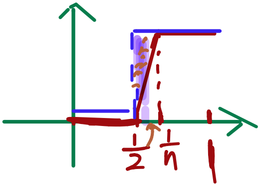
- $$\{f_n(x)\}_{n\geq 2} = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ n\left(x - \frac{1}{2}\right), & \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} < x \leq 1 \end{cases}$$

  It's a Cauchy sequence in $(\mathcal{C}[0,1], \|\cdot\|_1)$ since

  $$\|f_n - f_m\|_1 = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \to 0 \text{ as } n, m \to \infty$$

  Let

  $$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}$$

  $n \to \infty$

  Then

  $$\|f_n - f\|_1 = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

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  Then

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  However, $f \notin \mathcal{C}[0,1]$ and thus $(\mathcal{C}[0,1], \|\cdot\|_1)$ is not a Banach space.

End

See you next week